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The Menger-like Property of the Tree-width of Infinite Graphs*

IGOR KRÍŽ

*Department of Mathematics, The University of Chicago,
57345 University Avenue, Chicago, Illinois 60637*

AND

ROBIN THOMAS[†]

*School of Mathematics, Georgia Institute of Technology,
Atlanta, Georgia 30332*

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The tree-width of a (possibly infinite) graph G is the minimum n such that G may be decomposed into a “tree-structure” of pieces each with at most $n + 1$ vertices. We prove that if G has tree-width n , then G can be decomposed in such a way that for every $k \geq 0$ and any two pieces P and Q of this “tree-structure” either there are k disjoint paths between P and Q , or there is a third piece R between P and Q with at most $k - 1$ vertices. Our proof makes use of a finite version of this result proved by the second author and a compactness argument. © 1991 Academic Press, Inc.

1. INTRODUCTION

A graph is a pair $G = (V, E)$, where V is a (possibly infinite) set and E is a subset of $[V]^2$, the set of all 2-element subsets of V (thus we do not allow loops and multiple edges, but that is without loss of generality). We write $V(G) = V$ and $E(G) = E$. A *tree-decomposition* of a graph G is a pair (T, W) , where T is a tree and $W = (W_t : t \in V(T))$ is such that

$$(W1) \quad \bigcup_{t \in V(T)} W_t = V(G),$$

$$(W2) \quad \text{every edge } e \in E(G) \text{ has both its endpoints in some } W_t,$$

$$(W3) \quad W_t \cap W_{t''} \subseteq W_{t'} \text{ whenever } t' \text{ is on the path between } t \text{ and } t'' \text{ in } T.$$

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The *width* of the tree-decomposition (T, W) is $\sup_{t \in V(T)} |W_t| - 1$, provided that the sup is finite, otherwise it is undefined. A graph G has *tree-width* w if w is the least non-negative integer such that G admits a tree-decomposition of width w .

It can be shown that graphs with tree-width ≤ 1 are precisely the forests, graphs with tree-width ≤ 2 are the series-parallel ones. The complete graph K_n has tree-width $n - 1$ and the $n \times n$ grid (the adjacency graph of the $n \times n$ chessboard) has tree-width n .

The notion of tree-width was introduced by Robertson and Seymour [5] who used it in the well-quasi-ordering theory and in some algorithmic investigations. One of us, in order to extend the results of Robertson and Seymour on well-quasi-ordering finite graphs to the infinite case, proved the following two theorems (for the first one see [8], or [10] for a simplified proof).

(1.1) *If each finite subgraph of a graph G has tree-width $\leq w$ then G has tree-width $\leq w$.*

(1.2) *If G has tree-width w then it admits a tree-decomposition (T, W) satisfying (W1), (W2), (W3), and*

(W4) $|W_t| \leq w + 1$ for any $t \in V(T)$,

(W5) *for any $t_1, t_2 \in V(T)$, any $X_1 \subseteq W_{t_1}$, $X_2 \subseteq W_{t_2}$ and any $k > 0$ such that $|X_1| \geq k$ and $|X_2| \geq k$ either there are k disjoint paths in G , each between X_1 and X_2 , or there is a vertex t on the path in T between t_1 and t_2 such that $|W_t| < k$.*

Clearly (W4) says that the width of (T, W) is $\leq w$. For finite graphs the last theorem is proved in [7], but the original proof was unnecessarily complicated in the general case (and thus will not be published). The aim of this paper is to give a simple proof of (1.2). We use a compactness argument, but since there is a problem with direct application of compactness arguments we must find an equivalent locally verifiable property. For that we develop further Thomassen's idea [10] to use chordal graphs. Applications of (1.2) can be found in [6, 9].

We need the following two lemmas about tree-decompositions. The first is a lemma from [5], the proof of the second is left to the reader.

(1.3) *Let (T, W) be a tree-decomposition of a graph G , let $t_1, t_2 \in V(T)$, let P be a path in G between W_{t_1} and W_{t_2} , and let $t \in V(T)$ lie on the path in T between t_1 and t_2 . Then P uses a vertex of W_t .*

(1.4) *Let (T, W) be a tree-decomposition of a graph G and let K be a finite complete subgraph of G . Then there exists $t \in V(T)$ such that $V(K) \subseteq W_t$.*

A graph is called *chordal* if it contains no induced cycle of length ≥ 4 . We need the following lemma.

(1.5) *Let G be a chordal graph, let $k \geq 0$ be an integer, and let K, K' be complete subgraphs of G . Then the following statements are equivalent.*

- (i) *There are at least k disjoint paths in G between K and K' .*
- (ii) *There exists a sequence $K = K_1, \dots, K_n = K'$ of complete subgraphs of G such that $|V(K_i) \cap V(K_{i+1})| \geq k$ for $i = 1, \dots, n-1$.*

Proof. It is easily seen that if there exists a graph for which (1.5) fails, then there also exists a finite such graph. Hence it is enough to prove (1.5) for finite graphs, which is an easy exercise in chordal graphs and is left to the reader. ■

Finally, we require Rado Selection Lemma, as follows.

(1.6) *Let E be a set and for every finite set $A \subseteq E$, let $f_A: A \rightarrow \{0, 1\}$ be a mapping. Then there exists a mapping $f: E \rightarrow \{0, 1\}$ such that for every finite $A \subseteq E$ there exists a finite set B such that $A \subseteq B \subseteq E$ and $f(a) = f_B(a)$ for every $a \in A$.*

Proof. The system (f_A) forms a net in the topological space $\{0, 1\}^E$, which is compact by Tychonoff's theorem. Hence (see [2]) (f_A) has a cluster point, say f , which is the desired mapping. See [4] for a different proof. ■

2. MAIN RESULT

The following result can be deduced from Halin's theory of simplicial decompositions [1], but we prove it from first principles.

(2.1) *Let G be a chordal graph and $w \geq 0$ an integer such that G contains no isomorphic copy of K_w . Then G admits a tree-decomposition (T, W) such that for every $t \in V(T)$, W_t induces a complete graph in G .*

Proof. Let Δ be the complete graph whose vertices are the vertex-sets of maximal complete subgraphs of G . For an edge $\{K_1, K_2\}$ of Δ define its weight to be $|K_1 \cap K_2|$. We first construct a "maximum weight" spanning tree T of Δ . Let $E_{w-1} = \emptyset$ and assuming that we have already constructed $E_{w-1} \subseteq \dots \subseteq E_{i+1}$, let E_i be such that

- (a) E_i is obtained from E_{i+1} by adding a (possibly empty) set of edges of weight i ,

- (b) E_i does not contain a cycle, and
- (c) E_i is maximal subject to (a) and (b)

Since every edge of \mathcal{A} has weight $\leq w-2$, it follows that E_0 is the set of edges of a spanning tree T of \mathcal{A} . Let $W = \{t: t \in V(T)\}$.

- (1) (T, W) satisfies (W3).

Suppose for a contradiction that K_1, \dots, K_n ($n \geq 3$) are the vertices of a path in T such that $v \in K_1 \cap K_n - K_i$ for all $i = 2, \dots, n-1$ and for some $v \in V(G)$. Let $k = \min\{|K_i \cap K_{i+1}|: i = 1, \dots, n-1\}$ and let G' be the graph obtained from G by deleting v and all edges incident with v . We deduce from (1.5) applied to G' that there exist k disjoint paths in G' between $K_1 - \{v\}$ and $K_n - \{v\}$, and hence there exist $k+1$ disjoint paths in G between K_1 and K_n . By (1.5) applied to G there exists a path in \mathcal{A} between K_1 and K_n with all edges of weight $\geq k+1$, a contradiction to the maximality of T .

It follows from (1) that (T, W) is as desired. ■

Let G be a graph of tree-width w and let G be a subgraph of a chordal graph H . We say that H is an M -closure of G if H contains no isomorphic copy of K_{w+2} and for any two complete subgraphs K_1 and K_2 of H and any integer $k \geq 0$, if there are k disjoint paths in H between $V(K_1)$ and $V(K_2)$, then there are k disjoint paths in G between $V(K_1)$ and $V(K_2)$.

(2.2) *Let G be a graph of tree-width w . Then G has an M -closure if and only if it admits a tree-decomposition (T, W) satisfying (W1)–(W5).*

Proof. Assume first that G has a tree-decomposition (T, W) satisfying (W1)–(W5). Let H be the graph with vertex set $V(G)$ and with all edges of the form $\{u, v\}$ where $u, v \in W_t$ for some $t \in V(T)$. Then G is a subgraph of H by (W2), it is easy to see that H is chordal, and it follows from (1.4) that H contains no K_{w+2} . Let K_1 and K_2 be two complete subgraphs of H and $k \geq 0$ an integer such that there are k disjoint paths between $V(K_1)$ and $V(K_2)$ in H . By (1.4) there exist vertices $t_1, t_2 \in V(T)$ such that $V(K_1) \subseteq W_{t_1}$ and $V(K_2) \subseteq W_{t_2}$. By (W5) there exist k disjoint paths between $V(K_1)$ and $V(K_2)$ in G , because $|W_t| \geq k$ for every $t \in V(T)$ between t_1 and t_2 by (1.3).

For the converse let H be an M -closure of G , let (T, W) be the tree-decomposition of H as in (2.1). Then (T, W) is obviously a tree-decomposition of G and has width $\leq w$, because H contains no K_{w+2} . We subdivide each edge e of T by a vertex t_e , denote the tree thus obtained by R , and put

$$Y_r = \begin{cases} W_r & \text{if } r \in V(T) \\ W_{t_e} \cap W_{t'} & \text{if } r = t_e \text{ and } e = \{t, t'\} \end{cases}$$

and $Y = (Y_r: r \in V(R))$. Then (R, Y) is a tree-decomposition of G of width $\leq w$, we must verify that it satisfies (W5). Indeed, let $r_1, \dots, r_n \in V(R)$ be a path in R , let $X_i = Y_{r_i}$ ($i = 1, \dots, n$), let k be an integer, and let $X_0 \subseteq X_1$, $X_{n+1} \subseteq X_n$ be sets such that $|X_i| \geq k$ for $i = 0, \dots, n+1$. Then also $|X_i \cap X_{i+1}| \geq k$ by the definition of Y , and (since every X_i induces a complete graph in H) it follows from (1.5) that there exist k disjoint paths in H between X_0 and X_{n+1} , and hence there exist k disjoint paths in G between X_0 and X_{n+1} , as desired. ■

(2.3) *Every finite graph has an M -closure.*

Proof. Every finite graph of tree-width w admits a tree-decomposition (T, W) satisfying (W1)–(W5) by the finite version of (1.2) proved in [7]. Hence it has an M -closure by (2.2). ■

Let G be a graph and $A \subseteq [V(G)]^2$. We denote by $G \upharpoonright A$ the subgraph of G induced by all vertices which are elements of some set in A .

(2.4) *Every graph of finite tree-width has an M -closure.*

Proof. Let G be a graph of tree-width w , and let $E = [V(G)]^2$. For every finite set $A \subseteq E$ there exists by (2.3) an M -closure H_A of $G \upharpoonright A$. We define a mapping $f_A: A \rightarrow \{0, 1\}$ by

$$f_A(\{u, v\}) = \begin{cases} 1 & \text{if } u, v \text{ are adjacent in } H_A \\ 0 & \text{otherwise.} \end{cases}$$

Let $f: E \rightarrow \{0, 1\}$ be as in (1.6) and let H be the graph with vertex-set $V(G)$ and edges $\{u, v\}$ such that $f(\{u, v\}) = 1$. Then it is easy to verify that H is an M -closure of G . For instance, let K_1 and K_2 be two complete subgraphs of H and $k \geq 0$ an integer such that $V(K_1)$ and $V(K_2)$ are joined by k disjoint paths P_1, \dots, P_k in H . Then there exists a finite set $A \subseteq E$ such that $K_1, K_2, P_1, \dots, P_k$ are all subgraphs of H_A and since H_A is an M -closure of $G \upharpoonright A$ there exist k disjoint paths between $V(K_1)$ and $V(K_2)$ in $G \upharpoonright A$, and hence in G , as desired. ■

Proof of (1.2). Let G be a graph of tree-width w . Then G has an M -closure by (2.4) and thus G has a tree-decomposition satisfying (W1)–(W5) by (2.2). ■

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